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Simulating 2D Flows with Viscous Vortex Dynamics

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Approximate solutions of the two-dimensional Navier–Stokes equation can be constructed as a superposition of viscous Lamb vortices. Requiring minimum deviation from the Navier–Stokes equation, one gets a set of ordinary differential equations for the positions, strength and width of the vortices. We calculate the deviation of the solution from the Navier–Stokes equation in the square norm. The time dependence of this error is determined and discussed.

KEY WORDS: Hydrodynamics; Navier–Stokes equation; turbulence; Lamb vortex.

1. INTRODUCTION

We have pleasure in dedicating this paper to Prof. Gregoire Nicolis on the occasion of his 60th birthday.

Two dimensional flows often appear in the atmosphere due to stratification. As the typical horizontal extension of the flow is very large, the Reynolds number $\mathrm{Re} = UL/\nu$ is very large, too,[3] thus the flow is dominated by the inertial term $(\vec{v}\,\nabla)\,\vec{v}$ of the Navier–Stokes equation. The presence of the viscosity term is nevertheless important, as any decay of the flow is due to dissipation described by this term. Note that the effect of the viscosity is especially important at small length scales.

Therefore, there is an interest in methods that work in the regime of high Reynolds numbers.[(1)] Our specific motivation stems from atmospheric chemistry, where transport at long distances is determined by the large scale behavior of the flow, while chemical reactions take place locally.

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[3] U, L and ν stand for the characteristic velocity, length scale and the kinematic viscosity, respectively. In the atmosphere $\mathrm{Re} = 10^8 \cdots 10^{11}$.

Therefore, it is desirable to have a model for the background flow which is correct at both large and small length scales. In the present paper we discuss such a method.

Incompressibility in the two dimensional case allows of the introduction of the scalar stream function ψ whose derivatives determine the velocity by

$$v_x = \frac{\partial \psi}{\partial y} \tag{1}$$

$$v_y = -\frac{\partial \psi}{\partial x} \tag{2}$$

In terms of the vorticity

$$\omega = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \tag{3}$$

the Navier–Stokes equation reads

$$\frac{\partial \omega}{\partial t} + (\vec{v}\,\nabla)\,\omega = \nu\,\Delta\omega \tag{4}$$

In the nonviscous case (i.e., when $\nu = 0$ and thus Re becomes infinite) there exist a special class of exact solutions, called point vortex dynamics. Indeed, the stream function

$$\psi = \sum_j \Gamma_j \ln|\vec{r} - \vec{r}_j(t)| \tag{5}$$

(and the corresponding velocity field) satisfies Euler's equation exactly if

$$\dot{x}_j = \sum_{k \neq j} \Gamma_k \frac{y_j - y_k}{(\vec{r}_j - \vec{r}_k)^2} \tag{6}$$

$$\dot{y}_j = -\sum_{k \neq j} \Gamma_k \frac{x_j - x_k}{(\vec{r}_j - \vec{r}_k)^2} \tag{7}$$

Note that these equations can be derived from the Hamiltonian

$$H = \sum_j \sum_{k \neq j} \Gamma_j \Gamma_k \ln|\vec{r}_j - \vec{r}_k| \tag{8}$$

by the usual canonical formalism if the "coordinates" q_j and "momenta" p_j are defined by

$$q_j = \Gamma_j x_j \tag{9}$$

$$p_j = y_j \tag{10}$$

Equation (5) describes a superposition of point vortices

$$v_x = -\frac{y - y_j}{2\pi(\vec{r} - \vec{r}_j)^2} \tag{11}$$

$$v_y = \frac{x - x_j}{2\pi(\vec{r} - \vec{r}_j)^2} \tag{12}$$

thus Eq. (7) determines the motion of the vortex centres.

In the viscous case the point vortices are replaced by nonsingular ones[2, 3] in various ways. A further idea consists in representing the diffusional term as a stochastic process. This has been used for the exact solution of the two-vortex problem[4] and as an approximation for the many-vortex problem.[2] The convergence of the latter is limited by the error bars of stochasticity which decrease like $1/\sqrt{N}$ where N is the number of vortices. In view of this an alternative scheme has been suggested in which the Laplacian operator of the diffusive term is replaced by an integral operator that in turn is discretized.[5, 6] This scheme has been applied in a series of papers.[7, 8]

The one-point vortex solution has an exact generalization in the viscous case. This is the Lamb vortex,[9] given by

$$v_x = -\frac{y - y_0}{2\pi(\vec{r} - \vec{r}_0)^2}\left[1 - \exp\left(-\frac{(\vec{r} - \vec{r}_0)^2}{\delta^2}\right)\right] \tag{13}$$

$$v_y = \frac{x - x_0}{2\pi(\vec{r} - \vec{r}_0)^2}\left[1 - \exp\left(-\frac{(\vec{r} - \vec{r}_0)^2}{\delta^2}\right)\right] \tag{14}$$

$$\omega = \frac{1}{\pi\delta^2}\exp\left(-\frac{(\vec{r} - \vec{r}_0)^2}{\delta^2}\right) \tag{15}$$

where δ^2 depends linearly on time, namely,

$$\delta^2 = S + 4\nu t \tag{16}$$

It is easily seen indeed that in the nonviscous case ($\nu = 0$) and for $S \to 0$ the Lamb vortex goes over to the point vortex solution. As point vortex dynamics is exact in the nonviscous case, for large Reynolds numbers one expects that a superposition of Lamb vortices can be a good approximate solution of the Navier–Stokes equation (although an exact solution of this type does not exist). Therefore, we consider the Ansatz

$$\omega = \sum_j \frac{\Gamma_j(t)}{\pi(S_j(t)+4\nu t)} \exp\left(-\frac{(\vec{r}-\vec{r}_j(t))^2}{S_j(t)+4\nu t}\right) \tag{17}$$

for the vorticity and analogously for the velocity field. Such an Ansatz (with constant Γ_j and S_j) has been introduced in ref. 10. Further work along this line and applications can be found in refs. 11–12.

In contrast to the papers mentioned above the time dependence of the parameters is determined from the condition that the Ansatz be the "best" approximation to the Navier–Stokes equation. In order to define precisely in which sense the solution is best, we consider a variational approach.(10) Suppose in general, that an Ansatz with time-dependent parameters is given, that is, the spatial dependence of the stream function is

$$\psi = \psi(\vec{r}, a_1(t), a_2(t),..., a_n(t)) \tag{18}$$

When inserting this Ansatz into the Navier–Stokes equation, $\dot{a}_j$ appears in $\partial\omega/\partial t$ (and only there). Therefore, one can choose these time derivatives so as to minimize the deviation from the Navier–Stokes equation (for simplicity using the square integral norm), i.e., we require

$$\int d^2\vec{r} \left(\frac{\partial\omega}{\partial t} + (\vec{v}\,\nabla)\,\omega - \nu\,\Delta\omega\right)^2 = \text{minimum} \tag{19}$$

As

$$\frac{\partial\omega}{\partial t} = \sum_j \frac{\partial\omega}{\partial a_j}\,\dot{a}_j \tag{20}$$

minimization of Eq. (19) gives

$$\sum_k \int d^2\vec{r}\,\frac{\partial\omega}{\partial a_j}\,\frac{\partial\omega}{\partial a_k}\,\dot{a}_k = \int d^2\vec{r}\,\frac{\partial\omega}{\partial a_j}\,(\nu\,\Delta\omega - (\vec{v}\,\nabla)\,\omega) \tag{21}$$

This is a set of implicit quasilinear ordinary differential equations i.e., it has the structure

$$\sum_k M_{jk}\dot{a}_k = f_j \tag{22}$$

where the matrix M_{jk} and the vector f_j depend on the parameters a_i. Certainly, this method is generally applicable to find approximate solutions to the Navier–Stokes equation. An interesting example is (on a finite rectangular domain with periodic boundary conditions) when one uses the Ansatz

$$\psi = \sum_{\{\vec{k}\}} a_{\vec{k}}(t)\, \mathrm{e}^{i\vec{k}\vec{r}} \tag{23}$$

with a carefully chosen finite self-similar set of wave vectors $\vec{k}$, called the reduced wave vector set.[13–15] Note that this method gives account of many of the statistical properties of the turbulent flow correctly (except for small-scale intermittency), even at very high (10^8–10^9) Reynolds numbers.

We apply the variational method by using the Ansatz (17). The resulting equations will be given and discussed in the next section. These equations are slight generalizations of those given in ref. 10. We present them in a form which expresses their structure and symmetry rather clearly. In Section 3 we calculate the deviation from the Navier–Stokes equation explicitly. To our best knowledge, this has never been done before. Finally, we present some numerical results and conclude.

2. THE APPROXIMATE EQUATIONS

Inserting the Ansatz (17) into Eq. (21) we get after lengthy but elementary calculations

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{\Gamma_j \Gamma_k}{\pi^2 \delta_j^2 \delta_k^2} (\nabla_{\vec{r}_j} \circ \nabla_{\vec{r}_k} I_{jk})\, \dot{\vec{r}}_k + \sum_{k=1}^{n} \frac{\Gamma_j \Gamma_k}{4\pi^2 \delta_j^2 \delta_k^2} (\nabla_{\vec{r}_j} \Delta_{\vec{r}_k} I_{jk})\, \dot{S}_k \\
&\quad + \sum_{k=1}^{n} \frac{\Gamma_j}{\pi^2 \delta_j^2 \delta_k^2} (\nabla_{\vec{r}_j} I_{jk})\, \dot{\Gamma}_k \\
&\quad = \sum_{k=1}^{n} \sum_{l \neq k} \frac{\Gamma_j \Gamma_k \Gamma_l}{2\pi^3 \delta_j^2 \delta_k^2} \nabla_{\vec{r}_j} (\partial_{y_k} I_{jkl}^{x} - \partial_{x_k} I_{jkl}^{y})
\end{aligned} \tag{24}$$

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{\Gamma_j \Gamma_k}{4\pi^2 \delta_j^2 \delta_k^2} (\nabla_{\vec{r}_k} \Delta_{\vec{r}_j} I_{jk})\, \dot{\vec{r}}_k + \sum_{k=1}^{n} \frac{\Gamma_j \Gamma_k}{16\pi^2 \delta_j^2 \delta_k^2} (\Delta_{\vec{r}_j} \Delta_{\vec{r}_k} I_{jk})\, \dot{S}_k \\
&\quad + \sum_{k=1}^{n} \frac{\Gamma_j}{4\pi^2 \delta_j^2 \delta_k^2} (\Delta_{\vec{r}_j} I_{jk})\, \dot{\Gamma}_k \\
&\quad = \sum_{k=1}^{n} \sum_{l \neq k} \frac{\Gamma_j \Gamma_k \Gamma_l}{8\pi^3 \delta_j^2 \delta_k^2} \Delta_{\vec{r}_j} (\partial_{y_k} I_{jkl}^{x} - \partial_{x_k} I_{jkl}^{y})
\end{aligned} \tag{25}$$

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{\Gamma_k}{\pi^2 \delta_j^2 \delta_k^2} (\nabla_{\vec{r}_k} I_{jk})\, \dot{\vec{r}}_k + \sum_{k=1}^{n} \frac{\Gamma_k}{4\pi^2 \delta_j^2 \delta_k^2} (\Delta_{\vec{r}_k} I_{jk})\, \dot{S}_k + \sum_{k=1}^{n} \frac{1}{\pi^2 \delta_j^2 \delta_k^2} I_{jk} \dot{\Gamma}_k \\
&\quad = \sum_{k=1}^{n} \sum_{l \neq k} \frac{\Gamma_k \Gamma_l}{2\pi^3 \delta_j^2 \delta_k^2} (\partial_{y_k} I_{jkl}^{x} - \partial_{x_k} I_{jkl}^{y})
\end{aligned} \tag{26}$$

As one can see, all the coefficients are derivatives of the quantity

$$I_{jk} = \frac{\pi}{\frac{1}{\delta_j^2} + \frac{1}{\delta_k^2}} \exp\left(-\frac{(\vec{r}_j - \vec{r}_k)^2}{\delta_j^2 + \delta_k^2} \right) \tag{27}$$

and

$$\vec{I}_{jkl} = \pi \frac{\frac{\vec{r}_j - \vec{r}_l}{\delta_j^2} + \frac{\vec{r}_k - \vec{r}_l}{\delta_k^2}}{\left(\frac{\vec{r}_j - \vec{r}_l}{\delta_j^2} + \frac{\vec{r}_k - \vec{r}_l}{\delta_k^2} \right)^2} \left\{ \exp\left(-\frac{(\vec{r}_j - \vec{r}_k)^2}{\delta_j^2 + \delta_k^2} \right) \right.$$
$$\left. - \exp\left(-\frac{\frac{(\vec{r}_k - \vec{r}_l)^2}{\delta_k^2 \delta_l^2} + \frac{(\vec{r}_l - \vec{r}_j)^2}{\delta_l^2 \delta_j^2} + \frac{(\vec{r}_j - \vec{r}_k)^2}{\delta_j^2 \delta_k^2}}{\frac{1}{\delta_j^2} + \frac{1}{\delta_k^2} + \frac{1}{\delta_l^2}} \right) \right\} \tag{28}$$

Here

$$\delta_j^2 = S_j + 4\nu t \tag{29}$$

At the derivation we used a simpler method than that given in ref. 10, namely, instead of using polar coordinates and Bessel functions, we calculated integrals of the type

$$\int d^2\vec{r} \exp\left[-\frac{(\vec{r} - \vec{r}_k)^2}{\delta_k^2} \right] \frac{1 - \exp\left[-\frac{(\vec{r} - \vec{r}_l)^2}{\delta_l^2} \right]}{(\vec{r} - \vec{r}_l)^2} \tag{30}$$

by using the identity

$$\frac{1 - \exp\left[-\frac{(\vec{r} - \vec{r}_l)^2}{\delta_l^2} \right]}{(\vec{r} - \vec{r}_l)^2} = \int_0^{1/\delta_l^2} da \exp[-a(\vec{r} - \vec{r}_l)^2] \tag{31}$$

and changing the order of the integrations.

The Eqs. (24)–(26) are a set of ordinary differential equations for the vortex centres, strengths and widths. They correspond to a special approximate solution of the Navier–Stokes equation—the speciality being that the initial condition should be a superposition of Lamb vortices. This excludes the possible presence of an overall shear flow, for instance. The present formulation does not take into account possible external forces, either, e.g., the Coriolis force is also absent (although it is quite important at large scale in the atmosphere). Even after these simplifications we do not have an exact solution. Nevertheless, the method is the better the closer we are to the point vortex dynamics of the nonviscous case. Therefore, if the initial distances among the vortex centres are much larger than the widths and the Reynolds number is high, we may expect a high accuracy. In other words, at least for a class of special initial conditions, we have a method which is the better the higher the Reynolds number is.

3. DEVIATION FROM THE NAVIER–STOKES EQUATION

In simple situations one may use other methods for comparison in order to get an information about the quality of the approximation, however, at high Reynolds numbers brute force methods fail, thus we need a self-consistent characterization of the accuracy. A rather straightforward way is to calculate the deviation (19) itself and compare it with the square norm of the terms of the equation. Certainly, if their ratio becomes of order unity, the method is unreliable.

Using the general notation (22) for the quantities appearing in Eqs. (24)–(26), the deviation (19) can be written as

$$\int d^2\vec{r}\left(\frac{\partial\omega}{\partial t}+(\vec{v}\,\nabla)\,\omega-\nu\,\Delta\omega\right)^2=\int d^2\vec{r}((\vec{v}\,\nabla)\,\omega)^2-\sum_k \dot{a}_k f_k \tag{32}$$

We shall compare this quantity with the square integral of the inertial term (which is dominant at high Reynolds numbers), i.e., we characterize the error of the method by

$$1-\frac{\sum_k \dot{a}_k f_k}{\int d^2\vec{r}((\vec{v}\,\nabla)\,\omega)^2} \tag{33}$$

If it is small compared to unity, then the accuracy is high and the method gives reliable results. The only quantity in Eq. (33) we still need to calculate is the integral which is just the square integral of the inertial term. The calculation is again done by using the identity (31) and leads to

$$\int d^2\vec{r}((\vec{v}\,\nabla)\,\omega)^2=\sum_i\sum_{j\neq i}\sum_k\sum_{l\neq k}\frac{\Gamma_i\Gamma_j\Gamma_k\Gamma_l}{2\pi^3\delta_i^4\delta_k^4}(I_1+I_2) \tag{34}$$

Here

$$I_1=-\vec{r}_{lk}\vec{r}_{ij}\left(\int_{u_1}^{v_1}dz\,\frac{\exp(-z)}{(z-z_r)^2+z_i^2}-\int_{u_2}^{v_2}dz\,\frac{\exp(-z)}{(z-z_r)^2+z_i^2}\right) \tag{35}$$

and

$$I_2 = \frac{2}{g_0} \int_{u_1}^{v_1} dz \, \frac{(g_{11} z + g_{10})(g_{22} z^2 + g_{21} z + g_{20}) \exp(-z)}{[(z - z_r)^2 + z_i^2]^2} - \frac{2}{h_0} \int_{u_2}^{v_2} dz \, \frac{(h_{11} z + h_{10})(h_{22} z^2 + h_{21} z + h_{20}) \exp(-z)}{[(z - z_r)^2 + z_i^2]^2} \tag{36}$$

Equations (35), (36) contain the following shorthand notations:

$$\vec{r}_{ij} = \vec{r}_i - \vec{r}_j \tag{37}$$

$$u_1 = \frac{\dfrac{\vec{r}_{ik}^{\,2}}{\delta_i^2 \delta_k^2}}{\dfrac{1}{\delta_i^2} + \dfrac{1}{\delta_k^2}} \tag{38}$$

$$v_1 = \frac{\dfrac{\vec{r}_{ik}^{\,2}}{\delta_i^2 \delta_k^2} + \dfrac{\vec{r}_{ij}^{\,2}}{\delta_i^2 \delta_j^2} + \dfrac{\vec{r}_{jk}^{\,2}}{\delta_j^2 \delta_k^2}}{\dfrac{1}{\delta_i^2} + \dfrac{1}{\delta_j^2} + \dfrac{1}{\delta_k^2}} \tag{39}$$

$$u_2 = \frac{\dfrac{\vec{r}_{ik}^{\,2}}{\delta_i^2 \delta_k^2} + \dfrac{\vec{r}_{il}^{\,2}}{\delta_i^2 \delta_l^2} + \dfrac{\vec{r}_{lk}^{\,2}}{\delta_l^2 \delta_k^2}}{\dfrac{1}{\delta_i^2} + \dfrac{1}{\delta_l^2} + \dfrac{1}{\delta_k^2}} \tag{40}$$

$$v_2 = \frac{\dfrac{\vec{r}_{ik}^{\,2}}{\delta_i^2 \delta_k^2} + \dfrac{\vec{r}_{il}^{\,2}}{\delta_i^2 \delta_l^2} + \dfrac{\vec{r}_{lk}^{\,2}}{\delta_l^2 \delta_k^2} + \dfrac{\vec{r}_{ij}^{\,2}}{\delta_i^2 \delta_j^2} + \dfrac{\vec{r}_{jk}^{\,2}}{\delta_j^2 \delta_k^2} + \dfrac{\vec{r}_{lj}^{\,2}}{\delta_l^2 \delta_j^2}}{\dfrac{1}{\delta_i^2} + \dfrac{1}{\delta_j^2} + \dfrac{1}{\delta_k^2} + \dfrac{1}{\delta_l^2}} \tag{41}$$

$$z_r = \frac{\vec{r}_{ij} \vec{r}_{il}}{\delta_i^2} + \frac{\vec{r}_{kj} \vec{r}_{kl}}{\delta_k^2} \tag{42}$$

$$z_i = \left(\frac{\vec{r}_{ij}}{\delta_i^2} + \frac{\vec{r}_{kj}}{\delta_k^2} \right) \wedge \vec{r}_{lj} \tag{43}$$

$$g_0 = \left(\frac{\vec{r}_{ij}}{\delta_i^2} + \frac{\vec{r}_{kj}}{\delta_k^2}\right)^2 \tag{44}$$

$$g_{11} = \left(\frac{\vec{r}_{ij}}{\delta_i^2} + \frac{\vec{r}_{kj}}{\delta_k^2}\right) \wedge \vec{r}_{kl} \tag{45}$$

$$g_{10} = -\left[\frac{\vec{r}_{ik}^{\,2}}{\delta_i^2}\frac{\vec{r}_{kj}}{\delta_k^2} + \left(\frac{\vec{r}_{ij}^{\,2}}{\delta_i^2} + \frac{\vec{r}_{kj}^{\,2}}{\delta_k^2}\right)\frac{\vec{r}_{ik}}{\delta_i^2}\right] \wedge \vec{r}_{kl} \tag{46}$$

$$g_{22} = -\frac{\vec{r}_{kj} \wedge \vec{r}_{ij}}{\delta_k^2} \tag{47}$$

$$g_{21} = (2z_r - 1)\,\frac{\vec{r}_{kj}}{\delta_k^2} \wedge \vec{r}_{ij} \tag{48}$$

$$\begin{aligned} g_{20} = \Bigg\{ & -\left(\frac{\vec{r}_{ij}}{\delta_i^2} + \frac{\vec{r}_{kj}}{\delta_k^2}\right)^2 \vec{r}_{lj} + \Bigg[\left(\frac{\vec{r}_{ij}^{\,2}}{\delta_i^2} + \frac{\vec{r}_{kj}^{\,2}}{\delta_k^2}\right) - \left(\frac{\vec{r}_{ij}^{\,2}}{\delta_i^2} + \frac{\vec{r}_{kj}^{\,2}}{\delta_k^2}\right)^2 \\ & + 2\left(\frac{\vec{r}_{ij}^{\,2}}{\delta_i^2} + \frac{\vec{r}_{kj}^{\,2}}{\delta_k^2}\right)\left(\frac{\vec{r}_{ij}}{\delta_i^2} + \frac{\vec{r}_{kj}}{\delta_k^2}\right)\vec{r}_{lj} - \left(\frac{\vec{r}_{ij}}{\delta_i^2} + \frac{\vec{r}_{kj}}{\delta_k^2}\right)^2 \vec{r}_{lj}^{\,2}\Bigg]\frac{\vec{r}_{kj}}{\delta_k^2}\Bigg\} \wedge \vec{r}_{ij} \end{aligned} \tag{49}$$

$$h_0 = \left(\frac{\vec{r}_{ij}}{\delta_i^2} + \frac{\vec{r}_{kj}}{\delta_k^2} + \frac{\vec{r}_{lj}}{\delta_l^2}\right)^2 \tag{50}$$

$$h_{11} = \left(\frac{\vec{r}_{ij}}{\delta_i^2} + \frac{\vec{r}_{kj}}{\delta_k^2} + \frac{\vec{r}_{lj}}{\delta_l^2}\right) \wedge \vec{r}_{kl} \tag{51}$$

$$h_{10} = -\left[\left(\frac{\vec{r}_{ik}^{\,2}}{\delta_i^2\delta_k^2} + \frac{\vec{r}_{il}^{\,2}}{\delta_i^2\delta_l^2} + \frac{\vec{r}_{kl}^{\,2}}{\delta_k^2\delta_l^2}\right)\vec{r}_{kj} + \left(\frac{\vec{r}_{ij}^{\,2}}{\delta_i^2} + \frac{\vec{r}_{kj}^{\,2}}{\delta_k^2} + \frac{\vec{r}_{lj}^{\,2}}{\delta_l^2}\right)\frac{\vec{r}_{ik}}{\delta_i^2}\right] \wedge \vec{r}_{kl} \tag{52}$$

$$h_{22} = -\left(\frac{\vec{r}_{ij}}{\delta_i^2} + \frac{\vec{r}_{kj}}{\delta_k^2} + \frac{\vec{r}_{lj}}{\delta_l^2}\right) \wedge \vec{r}_{ij} \tag{53}$$

$$h_{21} = (2z_r - 1)\left(\frac{\vec{r}_{ij}}{\delta_i^2} + \frac{\vec{r}_{kj}}{\delta_k^2} + \frac{\vec{r}_{lj}}{\delta_l^2}\right) \wedge \vec{r}_{ij} \tag{54}$$

$$\begin{aligned} h_{20} = \Bigg\{ & -\left(\frac{\vec{r}_{ij}}{\delta_i^2} + \frac{\vec{r}_{kj}}{\delta_k^2} + \frac{\vec{r}_{lj}}{\delta_l^2}\right)^2 \vec{r}_{lj} + \Bigg[\left(\frac{\vec{r}_{ij}^{\,2}}{\delta_i^2} + \frac{\vec{r}_{kj}^{\,2}}{\delta_k^2} + \frac{\vec{r}_{lj}^{\,2}}{\delta^2{}_l}\right) \\ & - \left(\frac{\vec{r}_{ij}^{\,2}}{\delta_i^2} + \frac{\vec{r}_{kj}^{\,2}}{\delta_k^2} + \frac{\vec{r}_{lj}^{\,2}}{\delta_l^2}\right)^2 + 2\left(\frac{\vec{r}_{ij}^{\,2}}{\delta_i^2} + \frac{\vec{r}_{kj}^{\,2}}{\delta_k^2} + \frac{\vec{r}_{lj}^{\,2}}{\delta_l^2}\right)\left(\frac{\vec{r}_{ij}}{\delta_i^2} + \frac{\vec{r}_{kj}}{\delta_k^2} + \frac{\vec{r}_{lj}}{\delta_l^2}\right)\vec{r}_{lj} \\ & - \left(\frac{\vec{r}_{ij}}{\delta_i^2} + \frac{\vec{r}_{kj}}{\delta_k^2} + \frac{\vec{r}_{lj}}{\delta_l^2}\right)^2 \vec{r}_{lj}^{\,2}\Bigg]\left(\frac{\vec{r}_{ij}}{\delta_i^2} + \frac{\vec{r}_{kj}}{\delta_k^2} + \frac{\vec{r}_{lj}}{\delta_l^2}\right)\Bigg\} \wedge \vec{r}_{ij} \end{aligned} \tag{55}$$

Note that the integrals (35) and (36) can be expressed in terms of the exponential integrals $E_n(z)$ where n runs from 1 to 4 and the argument is usually complex.

4. CONCLUSION

In the two dimensional case (compared to the three dimensional situation) an additional approximately conserved quantity emerges (which is strictly conserved in the zero viscosity limit), the enstrophy, which is just the spatial integral of the square of the vorticity. This leads to a different qualitative picture about the energy cascade and the dissipation. While in three dimensions large eddies decay to smaller eddies, in two dimensional flows just oppositely, small eddies merge and form larger eddies. In our numerical simulations this generic merging has indeed always occured. We have used some superposition of Lamb vortices on the infinite plane as the initial condition and then solved Eqs. (24)–(26). The deviation (33) has also been calculated. Here we demonstrate the accuracy of the method in the simple case of two vortices. The vortices are initially at (20, 0) and (-20, 0), their initial widths are unity. The kinematic viscosity were chosen $\nu = 0.25$, and both vortex strengths Γ_i had the value 25. Figure 1 shows the error (33) as a function of time. As the vortices merge, the accuracy decreases.

In conclusion, we discussed a semi-analytical approximation scheme which yields a class of special solutions to the Navier–Stokes equation. The method is the better the higher the Reynolds number is. Compared to other

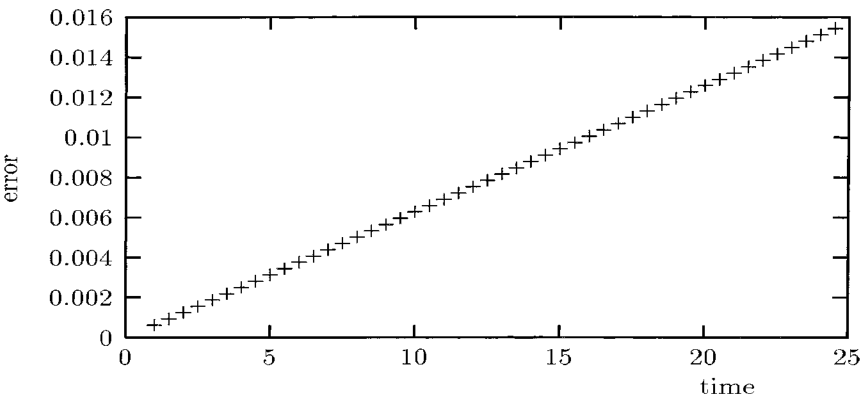

Fig. 1. Error (cf. (Eq. (33)) as a function of time for our approximation of a two vortex system. Initial center position of vortices: (-20, 0) and (20, 0), initial widths: 1, vortex strengths $\Gamma_i = 25$, kinematic viscosity: $\nu = 0.25$.

vortex approximations of the two dimensional Navier–Stokes our equations are not explicit. Instead we have to solve a set of implicit quasilinear differential equations. The CPU time for such problems increases in general with N^3 where N is the number of differential equations. By transforming the matrix in front to a sparse matrix we hope to decrease the needed CPU time considerably in the future. The distinct advantage of our procedure is that the equations are not found heuristically. On the contrary they are derived from a variational Ansatz. We calculated the square integral of the deviation from the Navier–Stokes equation for the first time, which allowed us quantifying the accuracy of the method.

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